Making Connections for Greater Base-Ten Understanding

This story is a part of the series:

***What's Next?* Stories of Teachers Engaging in Collaborative Inquiry Focused on Using Student Thinking to Inform Instructional Decisions**

What's Next?

Stories of teachers engaging in collaborative inquiry focused on using student thinking to inform instructional decisions

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Introduction

In the context of a professional-development experience, teachers gathered information about a group of second grade students' understanding of base-ten concepts through student interviews. Observing that many students understood the concepts of grouping and skip counting, the teachers created and implemented a classroom lesson intended to increase the use of strategies involving relational thinking. The lesson plan was also designed to engage students with each other's ideas. The teachers created a multiplication word problem, taking particular care to choose numbers that would encourage the use of students' existing knowledge of multidigit numbers to increase their understanding of the base-ten number system.

Relevant Florida Mathematics Standards

MAFS.2.NBT.2.5 Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.

MAFS.3.OA.1.3 Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.

Background Information

For further discussion of the mathematical concepts used in this lesson, consider reading chapters six, seven, and nine from Children's Mathematics: Cognitively Guided Instruction (Carpenter et al., 2015). Chapter six provides background information on children's understanding of base-ten number concepts and how multiplication and division problems can be used to develop this understanding. Chapter seven expands ways to increase base-ten concepts in the context of multidigit number problems. Chapter nine discusses ideas for creating a classroom environment designed to foster students' developmental understanding of base-ten concepts.

The lesson also places emphasis on counting strategies, especially skip counting, and organizing objects into groups of ten. The article "Counting collections" by Schwerdtfeger and Chan (2007) discusses the importance of guiding children's' counting processes to create the basis for more sophisticated number sense, such as counting groups of ten to solve multiplication with base-ten concepts.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (2015). *Children's Mathematics: Cognitively Guided Instruction (2nd. Ed.).* Portsmouth, NH: Heinemann.

Schwerdtfeger, J. K., & Chan, A. (2007). Counting collections. *Teaching Children Mathematics, 13*(7), 356–361.

Analyzing Student Thinking

A group of teachers interested in learning more about second-grade students' understanding of base-ten concepts and multidigit numbers interviewed a class of second graders with the goal of gaining insight into ways that grouping-type word problems (Carpenter et al., 2015) can be used to increase students' base-ten understanding. The one-on-one interviews lasted approximately 20 minutes, during which the interviewing teachers asked probing questions and took notes on how the students solved three word problems involving groups of items. The teachers had a set of alternative number pairs of varied difficulties for each of the interview problems and were free to replace the original numbers if they wanted to adjust the level of challenge for individual students. The interviewing teachers were instructed to choose smaller numbers from the alternative set if a child seemed to have trouble with the numbers in the problem. On the other hand, the interviewers were encouraged to use larger numbers if the child appeared to be comfortable with multidigit numbers. Each second-grade student was asked to solve the following three problems using strategies that made sense to them.

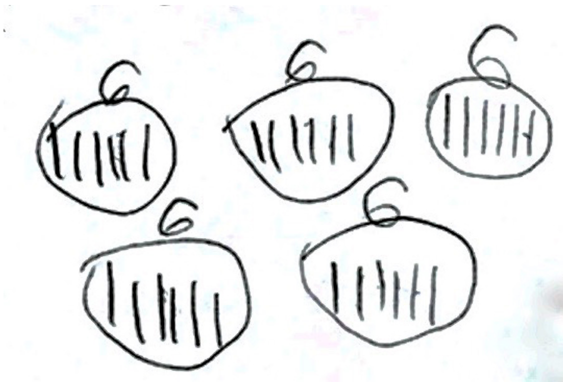

Figure 1. A student's drawing of a direct modeling—counts by ones *strategy for the bakery problem*

Problem A. *A bakery has 5 boxes of cookies with 6 cookies in each box. How many cookies are there? Alternative number sets: (3, 2), (6, 9), (8, 12)*

Problem B. *The store has 8 boxes of pencils with 10 pencils in each box. How many pencils does the store have? Alternative number sets: (4, 10), (14, 10)*

Problem C. *There are 30 students in the lunchroom. 10 students can sit at a table. How many tables are needed for all students to have a seat? Alternative number sets: (12, 3), (120, 10)*

The problems were posed one at a time, and each was written on its own sheet of paper. Students were encouraged to pay attention to the story context in the problem and to find their own way of solving the problem, using of any combination of mental strategies, paper/pencil, and physical manipulatives (e.g. linking cubes, base-ten blocks, fingers). Teachers refrained from helping students solve the problems and only asked questions to gain better understanding of the students' thinking processes.

After the interviews, the teachers analyzed the strategies used and organized the students into the following categories based on the strategies they used for each of the three interview problems[1].

1 The descriptions of strategies presented in this section are the current descriptions used by our team, and we consider them to be fluid, as our understanding of these ideas continues to evolve. For a more detailed discussion of these terms, consider reading Carpenter et al. (2015).

Strategies Typically Used by Students to Solve Problems Involving Multidigit Computation

The student who uses a *direct modeling—counts by ones* strategy uses objects to act out the story or situation in the order of events described in the story, representing each of the quantities in the problem as a set of ones using manipulatives or pictures and then counts the objects or pictures to determine the answer. For example, for the bakery problem (problem A), a student might create five groups with six cubes in each group and then count the cubes one at a time to find the total. Figure 1 shows an example of written work created by a student who used this strategy, counting each tick mark by ones to arrive at an answer of 30 cookies.

The student who uses a *direct modeling—counts by tens* strategy follows the story in the word problem and represents the multidigit numbers in the problem using manipulatives or pictures that reflect the base-ten structure of our number system (e.g., with base-ten blocks or base-ten pictures), then counts the objects or pictures to determine the answer. A student using a *direct modeling—counts by tens* strategy for the store problem (problem B), for example, would create eight groups of ten cubes and would count the cubes representing the pencils by tens, pointing at each group. Figure 2 offers an illustration of this strategy written on paper. This student pointed at each group and counted "10, 20, 30, 40, 50, 60, 70, 80."

In the lunchroom problem (problem C), a student using the *direct modeling—counts by tens* strategy might represent the students with groups of ten objects until there are 30 objects and might proceed to count by ten to keep track of the number of objects "10, 20, 30."

The bakery problem may prompt strategies involving skip counting, but the numbers in the problem do not fit well with skip counting by tens. Because the teachers thought skip counting might indicate a level of understanding more advanced than counting by ones, especially if the skip counting is related to the problem and

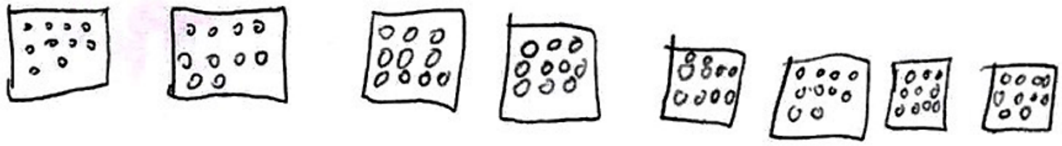

Figure 2. A student's drawing of a direct modeling—counts by tens *strategy for the store problem*

is done correctly, the teachers created a category of student strategies between *direct modeling—counting by ones* and *direct modeling—counting by tens* and called it *direct modeling—counting by groups*. A student using this strategy might create a drawing similar to the one in Figure 1 but might count by sixes, rather than tens, to find the total number of cookies.

The student who uses a *counting* strategy represents some of the quantities in the problem physically, but not all of them. Figure 3 shows the written work of a student who used this strategy for the bakery problem. This student drew five boxes and wrote a six in each box. Then, the student counted, "6, 12, 18, 24, 30" to determine the answer. Another example of a counting strategy for that problem might have involved a student's skip counting verbally by sixes and holding up one finger for each count, stopping after five fingers had been raised.

The teachers noted that counting by fives to solve the bakery problem would be a highly unusual strategy for a second-grade student, because very few students at this age have developed an understanding of the abstraction of the problem scenario into the mathematically true statement that $5 \times 6 = 6 \times 5$, or $a \times b = b \times a$ for all Real numbers a and b. Counting by sixes to solve this problem is perfectly acceptable for students at this age, and encouraging them to model the problem using the mathematics they do understand is desirable.

The student who uses a *fact recall—known facts* strategy responds that he or she "just knows" the answer. In the store problem, for example, the student would know that eight groups of ten make 80. The answer would be given very quickly.

The student who uses a *fact recall—derived fact*[2] strategy uses a known, related fact to help solve a problem involving an unknown fact. In the problem with 14 boxes of ten pencils, for example, the student might know that ten groups of ten is 100, and four more tens makes 140.

Strategies Used by Students in this Classroom

The names of students in the class were sorted according the strategies they used to solve each of the three problems during the interviews. Figures 4, 5, and 6 show the distribution of strategies used by students in the class for the three problems. The teachers noticed that many of the second graders grouped the objects and used skip counting. They also noticed that a few students counted by ones despite the large number of objects. On the other hand, a small minority of students knew the related number facts and used them to solve the problems. As Figure 6 also shows, several students did not successfully solve the division problem.

This classroom clearly included students who used a wide variety of strategies to solve prob-

2 Students may use many other strategies that involve *derived facts* while solving number-fact problems. For more information about *derived facts*, read chapter three of Carpenter et al. (2015).

Figure 3. A student's drawing of a counting *strategy for the bakery problem*

Direct modeling—counts by ones	*Direct modeling—counts by group*	*Counting—repeated addition*	*Fact recall—derived fact*	*Fact recall—known facts*
Janecia	Nadia	Brooklyn		Caleb
Jayda	James	Hannah		Andrea
Jerrylah	Cavon	Dariel		
	Samyra	Carolina		
	Deandre	Abriella		
		Landon		

Figure 4. The sorting of students by strategy used for the bakery problem

Direct modeling—counts by ones	*Direct modeling—counts by tens*	*Fact recall—derived fact*	*Fact recall—known facts*
Jayda	Janecia	Dariel	Caleb
Deandre	Abriella	Samyra	Andrea
	Nadia		Carolina
	James		
	Jerrylah		
	Landon		
	Cavon		

Figure 5. The sorting of students by strategy used for the store problem

Direct modeling—counts by ones	*Direct modeling—counts by tens*	*Fact recall—derived facts*	*Fact recall—known facts*	*Nonvalid*
Jayda	Abriella	Samyra	Caleb	Jerrylah
Nadia	James	Dariel		Janecia
	Cavon	Carolina		Hannah
	Landon			Andrea
				Deandre

Figure 6. The sorting of students by strategy used for the lunchroom problem

lems, indicating many different levels of mathematical understanding. Teachers noticed that the strategies students used were varied and complex enough to provide the foundation for a lesson that focused on examining relationships among number and operations. The teachers therefore developed the following overarching learning goal for the new lesson.

Students will begin to develop their understanding of the relationships among numbers and operations for multidigit numbers.

On the basis of how the students solved the three interview problems, the teachers set a number of more specific learning goals in support of the main goal. The teachers saw these goals to be connected to each other and to the students' development of base-ten understanding.

- Start using *fact recall—derived fact* strategies on multiplication facts
- Notice connections between multiplication and repeated addition
- Use *fact recall—derived fact* strategies on multiplication using facts of ten
- Develop strategies for nondecade numbers
- Facilitate precise mathematical communication that matches strategy use

Planning for the Lesson

Focusing on highlighting both the understanding that the base-ten number system is made up of groups of ten and that numbers and operations are related to each other in consistent ways, the teachers developed the following word problem for the new lesson.

There are 4 pens with 9 cows in each pen. How many cows are there?

Rationale for the selected problem

The teachers were aware that multiplication and division word problems can be used to help children develop their understanding of the base-ten number system by grouping objects into groups of ten. The chosen problem used groups of nine to address all the specific learning goals, including the goal to develop strategies for nondecade numbers. The number of items in each group was purposefully chosen to be very close to ten to increase the likelihood that some students would use known facts involving groups of ten to derive their answers.

Measurement-division problems—which give the total number of objects and the number of objects in each group and ask the problem solver to determine the unknown number in each group—are useful for introducing concepts of grouping by ten. The lunchroom problem above is an example. On the other hand, students tend to invent a wider variety of strategies for multiplication-grouping problems than they do for measurement-division problems. The teachers therefore created a multiplication word problem with a context familiar to the students and then proceeded to choose the numbers for the problem.

Because of the awareness of how much numbers affect the strategies that could be employed to solve a problem, the teachers spent a significant amount of time in choosing the ideal number set. They proposed several number pairs that were similar to those used in the interview and proceeded to consider the different ways related number facts could be used to perform the multiplication operation. They considered whether the proposed numbers would encourage the use of facts and whether they would meet the goals for the lesson. The following number pairs were considered (Figure 7).

Every number set provided the opportunity to explore the relationship between multiplication and repeated addition, so those related facts were not included in the analyses. The first set, four groups of ten, was discarded because these numbers would not provide students with opportunities to invent strategies for performing a multiplication operation with nondecade numbers. The next pair, three groups of six, had some interesting related facts that could be used but was not naturally related to the number ten. Four groups of eight, the third option, was well liked by the teachers, because the number eight was close enough to ten that a related fact of ten could be used to solve the problem. Another proposal of eight groups of seven was rejected, because it was thought to encourage the use of related doubles facts over related facts of tens, and encouraging students to use related double facts was not a goal for this lesson. The teachers decided to use the last number set, four groups of nine, because the number of items was very close to ten, therefore encouraging the students to consider using the ten and then compensating for it.

The teachers then anticipated various strategies students might use to solve this problem. They considered the several ways students who used a *direct modeling* strategy might illustrate it. In a group, they could draw three rows of three items, a row of five and a row of four items, a row of nine items, or even use a ten rod and cover one item. The teachers considered keeping track of the various illustrations and using them during the whole-class discussion to provide an opportunity to connect the models to the matching numerical notations and thus to more abstract concepts. Teachers also anticipated some students would use *counting* strategies without needing to represent every cow in the problem. Students using counting strategies could use repeated addition ($9 + 9 = 18$, $18 + 9 = 27$, $27 + 9 = 36$) or pairing groups and using repeated doubling ($9 + 9 = 18$, $18 + 18 = 36$) to solve this problem. These strategies can be used to help students to create connections between direct modeling strategies and

Number pairs (number of pens, number of cows)	*Related facts that might appear in students' strategies*		
(4, 10)	(4×10)	$(2 \times 10) + (2 \times 10)$	
(3, 6)	$(2 \times 6) + (1 \times 6)$	$(3 \times 5) + (3 \times 1)$	
(4, 8)	$(2 \times 8) + (2 \times 8)$	$(3 \times 8) + (1 \times 8)$	$(4 \times 10) - (4 \times 2)$
(8, 7)	$(4 \times 7) + (4 \times 7)$	$(5 \times 7) + (3 \times 7)$	$(7 \times 7) + (1 \times 7)$
(4, 9)	$(2 \times 9) + (2 \times 9)$	$(3 \times 9) + (1 \times 9)$	$(4 \times 10) - (4 \times 1)$

Figure 7. Possible number pairs and related facts that could be used to find the answer

The teachers decided to use the last number set, four groups of nine, because the number of items was very close to ten, therefore encouraging the students to consider using the ten and then compensating for it.

the more abstract relational thinking that uses multiplication concepts. They can also be used to help students to learn mathematical notation. The possibility that students would use *fact recall—known facts* was small, but they thought Caleb might know that four groups of nine are 36. Some students might use relational thinking that would be *fact recall—derived facts* strategies. Although the strategy seemed unlikely, they might know that three groups of nine make 27, so that the answer could be found by addition of another group of nine, a strategy related to repeated addition. They may also know that two groups of nine make 18, so two more groups would be 36, a strategy derived from grouping pairs. The teachers foresaw many ways to use related known facts but thought that the use of base-ten rods might introduce the related facts that four groups of ten are 40 and that taking away the excess four would provide the answer.

Strategies for differentiation to meet the needs of all students in the class

The teachers chose to use numbers small enough to be accessible to all students on the basis of the evidence from the interview. They hoped their choice of four groups of nine would allow the students who use a *direct modeling—counting by ones* strategy to solve the word problem with minimal counting errors. They also agreed that introducing mathematical notation was important even on the pictorial representations that would be discussed with the whole class. This strategy has the benefits of extending the mathematical thinking of students using *direct modeling* strategies, of facilitating the connection of mathematical concepts across strategies, and of addressing the goal of precise mathematical communication. The teacher decided to ask any students who solved the problem significantly faster than the rest of the class to solve the same problem another way, to keep them busy and engaged in relevant mathematics tasks.

In the spirit of advancing mathematical understanding in students on the basis of their own ideas, the teachers planned to revisit some of the strategies the students used before posing the new problem. This decision was specifically planned to encourage students to consider using one of the shared strategies when working on the subsequent problem. Teachers chose some strategies they considered to foster productive conversation and to highlight the understanding teachers wanted to disseminate. They chose a couple of *direct modeling* strategies (Jayda's and Jerrylah's) that would be used to introduce mathematical notation and additive and multiplicative numerical concepts. Next, the teachers chose a *counting* strategy using repeated addition (Landon's) and one that paired groups and added in chunks (Carolina's). Finally, they chose a strategy that demonstrated relational thinking (Samyra's) and understanding of base-ten concepts. Before the lesson, each of these students would be made aware of the plan to feature their work. They were told that the instructor would display their work and ask them to share their thinking with the class at the beginning of the lesson.

While the students worked, the teachers planned to watch for students who used any of the strategies shared during the whole-class discussion. The instructor prepared a list of the strategies students used on the bakery and store problems during the interview to help them in noticing possible adoptions of new strategies.

Lesson Plan

The following lesson took place in a second-grade classroom. The lesson was developed to align with the following overarching goal and the more specific learning goals derived from it.

> *Students will begin to develop their understanding of the relationships among numbers and operations for multidigit numbers.*

- Start using *fact recall—derived fact* strategies on multiplication facts
- Notice connections between multiplication and repeated addition
- Use *fact recall—derived fact* strategies on multiplication using facts of ten
- Develop strategies for nondecade numbers
- Facilitate precise mathematical communication that matches strategy use

1. The teacher began by saying to the class, "I want to learn from you how you thought about the problems you solved earlier. I am going to display some interesting thinking, and I am going to ask the child whose mind it came from to come and share his or her strategy with us, so we can learn."

2. The teacher read the bakery problem to the class, "A bakery has 5 boxes of cookies with 6 cookies in each box. How many cookies are there?"

3. The teacher displayed Jayda's strategy so every student could see it. She asked Jayda to come to the board to show the rest of the class how she solved the problem. (Jayda's work is shown in Figure 1.)

   a. The teacher told the class, "I want the rest of you to think about how Jayda solved the problem. I want you to think about how her strategy is different from yours."

   b. The teacher then asked, "Jayda, where are your boxes?"

   c. Jayda pointed at each of the five circles with tick marks as she counted, "1, 2, 3, 4, 5."

   d. The teacher asked Jayda, "How did you find the answer of 30?

   e. Jayda answered, "I counted, 1, 2, 3, 4, 5, 6, 7, 8, 9, (counts by ones),…, 30."

   f. The teacher asked," Is there any other way to count the cookies?"

   g. Jayda hesitated for a long time, so the teacher turned to Jerrylah, "Jerrylah, what do you think? Come to the board and show us what you did."

4. Jerrylah came to the board as the teacher displayed her strategy, shown in Figure 8.

a. Jerrylah pointed to the circled bars and explained, "I knew two boxes is six and six, that's 12." The teacher wrote under the two circled bars, 6 + 6 = 12. Jerrylah continued, pointing to the other three bars, "And three boxes is six and six and six, that's 18. So, the total is 12 plus 18 is 30." The teacher wrote under the other three bars, 6 + 6 + 6 = 18 and underneath that, 12 + 18 = 30.

b. The teacher asked the class "So you mean to tell me that five boxes with six cookies makes 30 cookies?" As she said this, the teacher wrote 5 x 6 = 30.

c. The students answered in the affirmative.

d. The teacher asked Jayda, "Where are the 12 and the 18 in your model?"

e. Jayda made a circle with her finger over the first two boxes in her drawing and said "12", then did the same to the other third boxes and said, "18." The teacher thanked Jayda and Jerrylah for their explanations.

5. The teacher turned to Carolina and asked her to come to the board to share her thinking on the same problem. Figure 8 below illustrates Carolina's strategy with the teacher's notation, added during this conversation. "Carolina solved the bakery problem with different numbers. You had eight boxes with 12 cookies in each box. Explain to us what you did."

a. Carolina explained, "I wrote eight 12s so I could remember how many I have. I added two 12s, and that's 24."

b. The teacher interjected, "How do you know it is 24?"

c. Carolina said, "Because two and two is four and ten and ten is 20, so that is 24. So, 24 and 24 is 48 (pointing at the first two vertical computations)."

d. The teacher interjected again, "How do you know that is 48?"

e. Carolina answered, "Four and four is eight and 20 and 20 is 40, so 48."

f. The teacher asked, "You originally had eight groups of 12." She numbered the eight 12's from one to eight, as seen on the left side of Figure 9. "Did you turn them all into 24?"

g. Carolina nodded in the affirmative.

h. The teacher continued, "So, how many 24s does that make?"

i. Carolina answered, after counting, "four 24s."

j. The teacher circled the four 24s, then said, "This is interesting. These eight twelves are also four 24s." She wrote 8 × 12 = 4 × 24 underneath (not seen in the figure). "And what did those 24s become?"

k. Carolina answered, "48."

l. The teacher prompted, "How many 48's are there?"

m. Carolina said, "Two," and the teacher circled the two 48 answers in the strategy.

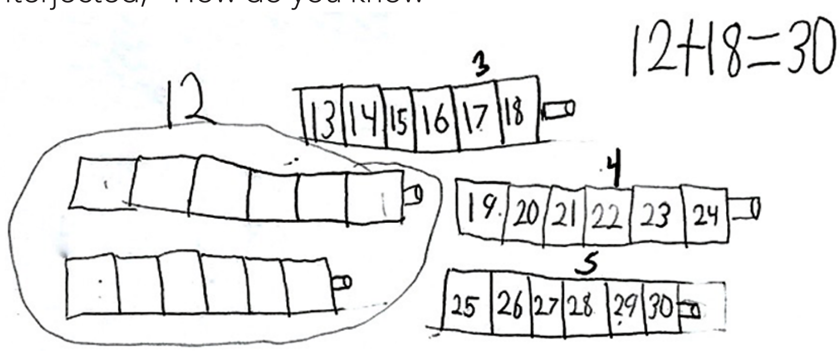

Figure 8. Jerrylah's student work for the bakery problem

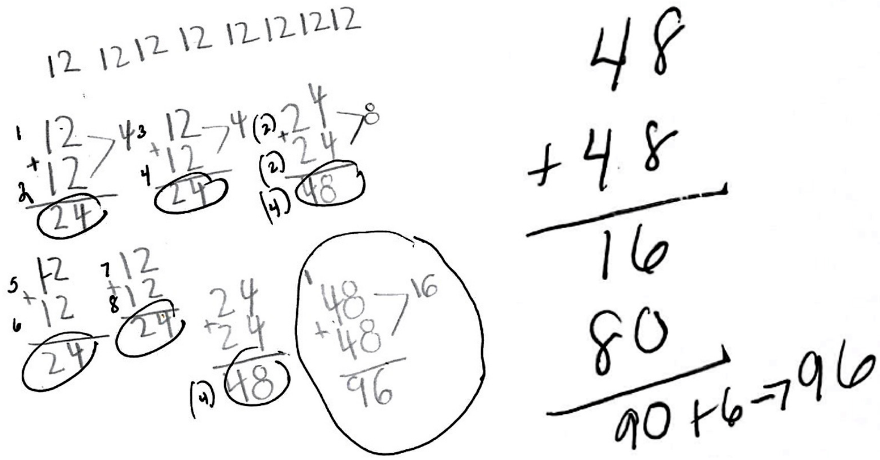

Figure 9. Carolina's written notation of her counting strategy for the bakery problem with additional notation provided by the teacher

n. The teacher exclaimed, "Wow! Eight groups of 12 became four groups of 24, then became two groups of 48!" She then added another expression to the previous equation, 8 × 12 = 4 × 24 = 2 × 48.

o. The teacher asked next from the entire class, "And what is two groups of 48?"

p. Students replied, "96," noticing the answer on Carolina's student work.

q. The teacher asked, "How do you know that?" After a few seconds of silence, the teacher wrote 48 + 48 vertically on the board and asked the class how to find the sum.

r. One student responded, "Eight and eight is 16, so write a six under the line and add a one above the four."

s. The teacher said, "If eight plus eight is 16, then we should write that under the line. What should we add next?"

t. Another student offered, "Four and four is eight."

u. The teacher asked, "Is that really a four?"

v. The student corrected himself, "40 and 40 is 80."

w. The teacher wrote 80 underneath 16, then asked for the next step.

x. A third student offered that ten and 80 make 90, and six more is 96, the answer.

6. The teacher showed another counting strategy for the same problem with eight groups of 12, this time Landon's work. She asked Landon to clarify his thinking for his classmates.

a. Landon explained, "I have eight groups of 12, so I added each 12 one at a time. I started with 12 and I added eight to it…"

b. The teacher asked for a clarification, "Where did you get the eight you added?"

c. Landon said, "Well, I'm adding the twelves broken up to make friendly numbers. So, I break up the second 12 into eight and four. I added the eight and got 20, my friendly number. Then I added the four and got 24." Landon's student work is shown in Figure 10.

d. Landon continued, "Then I broke the third 12 into six and six and added a six to the 24 to get to the next friendly number, 30. Then I added the other six and got 36. The next number I added was four to get 40, so I had to add another eight from the fourth 12."

e. The teacher asked, "What did you break up the fifth 12 into?"

f. Landon answered, "I broke it up into two and eight."

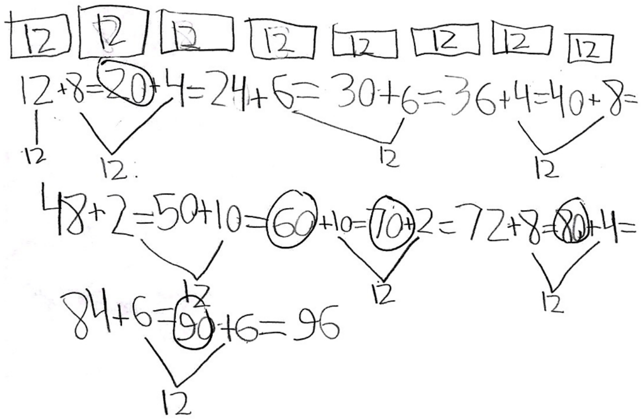

Figure 10. Landon's written notation for the repeated addition counting strategy he used for the bakery problem

g. "Why did you do that?" asked the teacher.

h. Landon said, "I had 48 so far, so I needed two more to get to 50. Then I had another ten left that I added and got 60."

i. The teacher said, "Wow, you used all those facts about 12!" She turned to the rest of the students and asked them, "Did you follow what Landon did?"

j. Some students answered in the affirmative, some did not answer. The teacher decided to go through the strategy again for clarification. During the second time through the explanation, several students seemed to better understand the repeated decomposition of 12 and even started giving the teacher the next steps.

7. The teacher next reminded students of the store problem and asked Samyra to explain the thinking she used to solve the problem with larger numbers, "The store has 14 boxes of pencils with ten pencils in each box. How many pencils does the store have?"

   a. Samyra explained to her peers, "I knew that ten tens is 100, and the four more tens is 40. I added them together, and it was 140."

   b. The teacher wrote on the board the following equations that matched Samyra's explanation: $(10 \times 10) + (4 \times 10) = 14 \times 10$
   $100 + 40 = 140$

   c. The teacher asked Samyra, "So, ten tens and four tens is the same as 14 tens?"

   d. Samyra answered in the affirmative. The teacher thanked the students for all the wonderful thinking they shared with each other.

8. In the second part of the lesson, the teacher told the students, "You have heard a bunch of great strategies so far. I would like you to solve a problem using any strategy that makes sense to you. Please solve the following problem,

There are 4 pens and there are 9 cows in each pen. How many cows are there?

9. The teacher asked students to solve the problem individually. While they worked on the problem, the teacher walked among students and took note of the strategies they used. She sometimes stopped to ask clarifying questions about a student's strategy. The teacher made notes of students who were making connections to some concepts explained earlier. She was looking specifically for students who were using a more advanced strategy than they had during the interview or some form of relational thinking. The teacher let the selected students know that she would ask them to share their strategy with the entire class. She arranged the strategies to be shared from the simpler strategies to the more sophisticated or abstract.

10. The teacher asked Janecia to share her strategy for the cows problem first. The teacher displayed Janecia's model, as seen in Figure 11.

   a. Janecia said, "Here are the pens and here are the nine cows in each pen. I counted them all, and there's 36."

   b. The teacher said, "Wow, your cows are very organized. I see you have them three in a row. I'm going to number them. 3, 6, 9,…, 36" , she said as she wrote the numbers on the sheet, as seen in Figure 11.

c. The teacher continued, "So you found, Janecia, that four pens with nine cows in each pen is 36 cows." She wrote 4 × 9 = 36 on the board.

11. The teacher addressed the whole class and asked, "How else could you count that?"

a. Nadia raised her hand and replied, "She could have counted ten minus one. 10, 20, 30, 40, then you would minus four."

b. The teacher asked, "Why minus four? Where did you get the ten from?"

c. Nadia answered, "I modeled with rods, and that's how I did it."

d. The teacher said, "Are you saying that you used a ten block and you took one away? Can you show me what that looks like?"

e. Nadia came to the front with her linking cubes and showed the class what she did. She brought four sticks of ten connected cubes, then removed one cube from each stick as seen in Figure 12.

f. Nadia explained, "So, I had four groups of ten. But I needed nine in each group, so I took one cube away from each. So, that's why you have 40 minus four."

g. The teacher said, "I see now. So, you started out with ten items in each group." The teacher wrote 10, 20, 30, 40 under Janecia's model's groups. "And then you took one away from each group." She wrote –1 above each group of objects.

h. The teacher continued, "So, you could say that four groups of nine is the same as four tens minus four ones". She wrote 4 × 9 = (4 × 10) – (4 × 1).

12. The teacher asked again, referring to Janecia's picture, "How else could you count that?"

a. After a few seconds of thinking, Carolina answered, "Three, six, nine,… you could use three threes to equal nine."

b. "How?" the teacher asked.

c. Carolina answered, "Nine, 12, 15,18, 21, 24, 27, 30, 33, 36."

d. The teacher asked the other students, "How many threes did Carolina count? Who's got their thinking hat on?"

e. Janecia, who was still at the board, answered, "12."

f. "Show me", the teacher said.

g. Janecia counted each row of circles.

h. The teacher said, "Are you saying to me that four groups of nine is the same as 12 groups of three?

i. Janecia answered in the affirmative and then the teacher wrote on the board, 4 × 9 = 12 × 3. "That is very interesting!"

13. The teacher next asked Dariel to show his work and explain his thinking. Figure 13 shows his strategy.

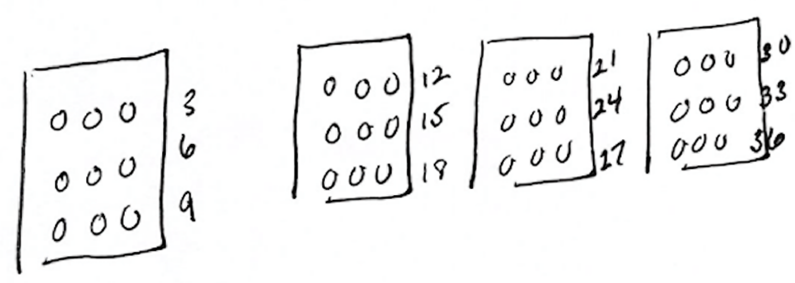

Figure 11. Janecia's direct modeling *strategy for the cows problem*

a. Dariel explained, "So I knew I had four groups of nine. I knew that is four times nine. I drew out the four groups and wrote the nine for each one."

b. The teacher asked, "Can you tell me what are those fours written in the squares?"

c. Dariel said, "Those are my four pens. I just wrote that so I know they're the four pens."

d. The teacher asked, "So, how did you find the total?"

e. Dariel answered, "I added two of the nines and that is 18. Then I added the other two nines."

f. The teacher asked, "How did you know that?"

g. Dariel said, "I just knew that. Nine plus nine is 18. Then I added the two 18's together. Ten and ten is 20, and eight and eight is 16. So, 20 plus 16 is 36."

14. The teacher told the class, "We have time for one more strategy. Carolina, will you please show us how you solved it?"

a. Carolina came to the front and showed her work, which appears in Figure 14.

b. Carolina said, "I did it two ways. I did it like Dariel here (pointing to the upper left section of her paper). Nine plus nine is 18. And 18 plus 18 is 36."

c. The teacher asked, "How did you know to add another 18?"

d. Carolina answered, "I knew there are four nines. So, the first 18 has two nines in it, and the second 18 has the other two nines."

e. The teacher said, "I see" and wrote a two in parenthesis next to each 18. What about the second way?"

f. Carolina explained, "I added nine and nine and that was 18. Then I added another two to get 20."

g. The teacher asked, "Why did you add a two?"

h. Carolina explained, "To get to 20. Then, I added the seven left from nine and got 27. I added three to get to 30 and then there was another six left from the last nine. So, my answer was again 36."

15. The teacher thanked everyone for the great thinking they displayed and ended the lesson.

Reflection

At the beginning of the lesson, five students explained the strategies they used to solve the interview problems. Some important mathematical ideas were showcased in those strategies, and the teachers wanted to expose the whole class to them. This process took approximately 30 minutes, however—much longer than the teachers thought it would. The time available for the students to work and share their thinking on the new problem was therefore limited. The teacher noted that, in future lessons, the initial sharing session should be limited to at most 20 minutes.

The teachers' overall impression was that the students really enjoyed the lesson. A lot of evidence showed that students were engaged in the lesson and learning from each other. The students were paying attention to one another's strategies and were able to answer teacher's questions about details from other students' work. They were also willing to try new strategies. In particular, Carolina, who shared her thinking both during the initial discussion time and during the problem's discussion, used Landon's strategy as her second way to solve the problem. The teachers noticed that several students had solved the problem two ways. The students also showed a great deal of perseverance in working on ways to model and solve the problem.

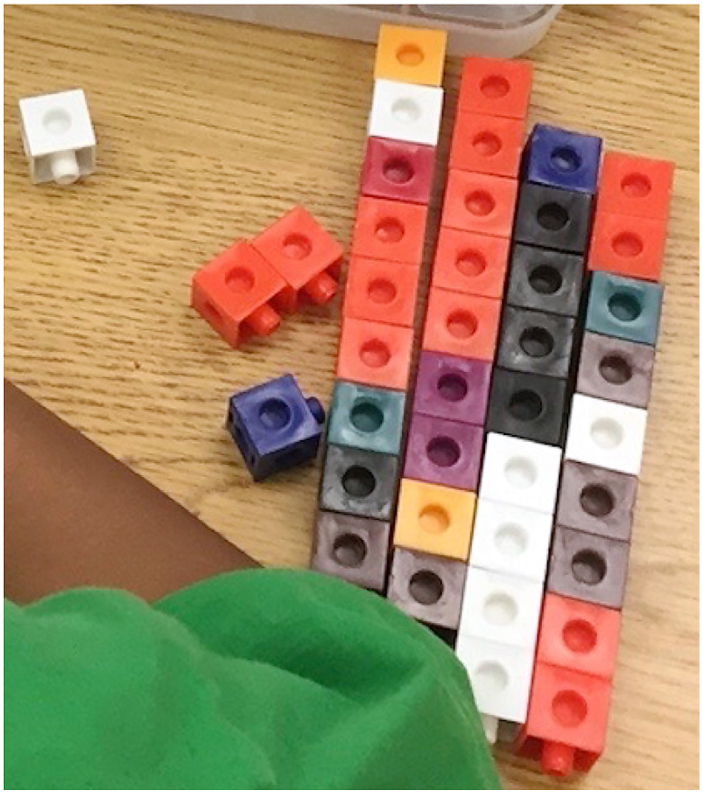

Figure 12. Nadia's direct modeling *strategy for the cows problem using connected cubes*

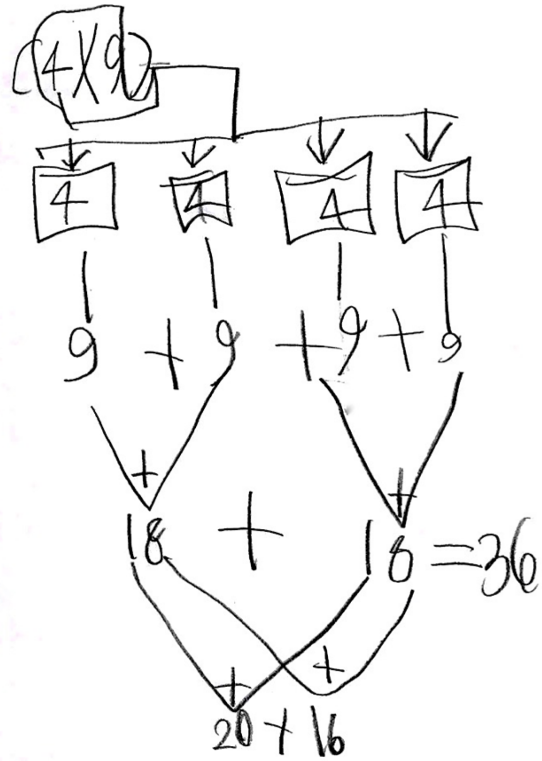

Figure 13. Dariel's strategy for the cows problem

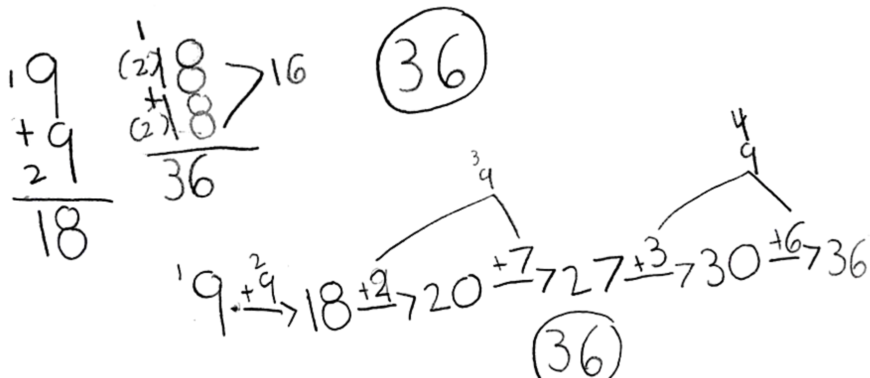

Figure 14. Carolina's two strategies for the cows problem

One of the major conclusions from this lesson was the importance of teacher questions and how they shape the things the children learn. The teachers admitted the importance of continuing to ask rigorous mathematical questions and to help others make sense of the strategies shared using them. The teachers thought the instructor's emphasis on writing down strategies using mathematical equations and connecting them to strategies through questions was also important. The observing teachers noticed that some students adopted the practice of writing down equations in their subsequent work. These students tried to be more mathematically consistent in their written explanations throughout the lesson.

Next steps for these students might involve lessons with groups of more than ten items, leading them to practice understanding of tens and decomposition of numbers for easy addition. Another possible direction could involve problems with three-digit numbers as answers.

What's Next?

Stories of teachers engaging in collaborative inquiry focused on using student thinking to inform instructional decisions

What's Next? is a collection of stories documenting professional development experiences shared by elementary teachers working collaboratively to study the complex process of teaching and learning mathematics. Each story in the collection describes practicing teachers studying the thinking processes of real students and using what they learn about those students to make decisions and try to help advance those students' understanding on that day.

The teachers in each story start by learning about how individual students are solving a set of mathematics problems. They use this freshly gathered knowledge of student thinking to develop near-term learning goals for students and a lesson plan tailored to specific students on that specific day. One of the teachers implements the planned lesson while the other teachers observe in real time. The teachers then gather to discuss and reflect on their observations and insights.

In these lessons, the practice of teaching is slowed way down. The stories tell of teachers who are studying student thinking and using that information to plan and implement instructional decisions at a pace that is much slower than it occurs in daily practice. The stories in this collection also depict many aspects in common with formative assessment and lesson study, both of which are a process and not an outcome.

The stories depict real situations that occurred in real time and include both successes and shortcomings. We hope that the stories may be studied and discussed by interested educators so that the lessons and ideas experiences of these teachers and instructional coaches may contribute to additional learning and sharing among other interested teachers.

Learn more about these and other stories at **http://www.teachingisproblemsolving.org/**